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A Lattice Green Function Introduction

Stefan Hollos*

Exstrom Laboratories LLC, 662 Nelson Park Dr, Longmont, Colorado 80503, USA

Abstract

We present an introduction to lattice Green functions.

*Electronic address: stefan@exstrom.com; URL: http://www.exstrom.com/stefan.html.

I. INTRODUCTION

This is a short concise introduction to the concept of a lattice Green function (LGF). The LGF is the discrete space counterpart to the more familiar continuous space Green function that has become such a versatile tool in many areas of theoretical physics. Some familiarity with the more common uses of Green functions, such as in the solution of partial differential equations, is helpful in what follows but not strictly necessary. Excellent introductions to Green functions can be found in Barton [1], Duffy [2], and Economou [3]. The LGF is often used in condensed matter, and statistical physics (random walk theory). A good discussion of some uses of the LGF can be found in Cserti [4]. It also appears, although most often not by name, when the finite difference approximation is used to solve partial differential equations. It is through this application that we will introduce the concept of a LGF. In particular we will use the LGF to show how the discretized Poisson equation can be solved in an infinite cubic (3 dimensional) and square (2 dimensional) lattice. It is perhaps appropriate to introduce the LGF in this way since solving the Poisson equation was George Green's original motivation for developing his eponymous functions [5]. A great deal of research has been done on lattice Green functions over the last fifty years or so and other introductions do exist, see for example Katsura et al [6] and the two recent papers by Cserti [4, 7]. The hope is that the simple examples given in this introduction will be accessible to the widest possible audience. The only knowledge assumed on the part of the reader is some familiarity with Dirac vector space notation and an understanding of eigenvalues and eigenvectors.

II. THREE DIMENSIONAL DISCRETE POISSON EQUATION

For the cubic lattice let $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$ be a set of orthogonal unit vectors, so that $\hat{x}_i \cdot \hat{x}_j = \delta(i,j)$. If the lattice spacing is a then the primitive lattice vectors are $\vec{a}_i = a\hat{x}_i$ and all points in the lattice are given by the lattice vectors

$$\vec{r}_n = n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3 \qquad n_i = \text{integer} \tag{1}$$

Using this notation, the Poisson equation on a cubic lattice takes the following form.

$$\sum_{i=1}^{3} [\phi(\vec{r}_n + \vec{a}_i) - 2\phi(\vec{r}_n) + \phi(\vec{r}_n - \vec{a}_i)] = f(\vec{r}_n) \tag{2}$$

We will refer to this as the discrete Poisson equation or DPE from here on. To fully define the equation, the size of the lattice and the boundary conditions need to be specified. These are however not important for the current discussion and will be specified later. In the limit as the lattice spacing goes to zero this becomes the continuous Poisson equation.

$$\sum_{i=1}^{3} \frac{\partial^2 \phi(\vec{r})}{\partial x_i^2} = g(\vec{r}) \tag{3}$$

Eq. 2 can be regarded as a finite difference approximation of eq. 3 with $f(\vec{r}_n) = a^2 g(\vec{r_n})$.

Much of the following development will be in terms of Dirac vector space notation. In this notation the DPE is

$$L|\phi\rangle = |f\rangle \tag{4}$$

Here L denotes the lattice Laplacian operator.

If we let $|n\rangle$ denote the lattice basis vector associated with the lattice point $\vec{r}_n$ then $\langle n|\phi\rangle = \phi(\vec{r}_n)$ and $\langle n|f\rangle = f(\vec{r}_n)$. In the lattice basis, the vectors $|\phi\rangle$ and $|f\rangle$ are

$$|\phi\rangle = \sum_n |n\rangle\langle n|\phi\rangle = \sum_n \phi(\vec{r}_n)|n\rangle \tag{5}$$

$$|f\rangle = \sum_n |n\rangle\langle n|f\rangle = \sum_n f(\vec{r}_n)|n\rangle \tag{6}$$

and eq. 4 is

$$\sum_n \langle l|L|n\rangle\langle n|\phi\rangle = \langle l|f\rangle \tag{7}$$

In terms of matrix and vector elements this becomes

$$\sum_n L_{ln}\phi(\vec{r}_n) = f(\vec{r}_l) \tag{8}$$

The matrix elements L_{ln} can be identified by comparing eq. 8 with eq. 2.

$$L_{ln} = \begin{cases} -6\delta(l,n) & \\ 1 & \text{if } |\vec{r}_n - \vec{r}_l| = a \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

This can also be expressed as follows.

$$L_{ln} = -6\delta(\vec{r_l},\vec{r}_n) + \sum_{i=1}^{3} [\delta(\vec{r}_l + \vec{a}_i, \vec{r}_n) + \delta(\vec{r}_l - \vec{a}_i, \vec{r}_n)] \tag{10}$$

Now we want to solve eq. 4 for $|\phi\rangle$. At least formally, the solution is

$$|\phi\rangle = L^{-1}|f\rangle \tag{11}$$

The problem therefore involves finding $L^{-1} = -G$, where the operator G is what we will call the lattice Green function. We can get an expression for the matrix elements of G by using an eigenvector expansion. It is easy to show that G and L have the same eigenvectors, and if λ is an eigenvalue of L then $-1/\lambda$ is an eigenvalue of G. The first step then is to find the eigenvalues and eigenvectors of L.

Until now, no assumptions have been made about the size of the lattice or the boundary conditions. We begin by assuming a finite lattice with $N_i$points in the direction $\vec{a}_i$and periodic boundary conditions. Periodic boundary conditions mean that for any lattice function $v(\vec{r}_n)$, the following will be true

$$v(\vec{r}_n + N_i\vec{a}_i) = v(\vec{r}_n) \qquad i = 1,2,3 \tag{12}$$

With these assumptions the eigenvalue problem for L can now be solved. Write the eigenvalue equation for L as follows.

$$L|v_m\rangle = \lambda_m|v_m\rangle \tag{13}$$

In the lattice basis the eigenvalue equation is

$$\sum_n \langle l|L|n\rangle\langle n|v_m\rangle = \lambda_m\langle l|v_m\rangle \tag{14}$$

or in terms of matrix elements

$$\sum_n L_{ln}v_m(\vec{r}_n) = \lambda_m v_m(\vec{r}_l) \tag{15}$$

Using eq. 9 for the matrix elements of L, eq. 15 becomes.

$$v_m(\vec{r}_l+\vec{a}_1)+v_m(\vec{r}_l-\vec{a}_1)+v_m(\vec{r}_l+\vec{a}_2)+v_m(\vec{r}_l-\vec{a}_2)+v_m(\vec{r}_l+\vec{a}_3)+v_m(\vec{r}_l-\vec{a}_3)-6v_m(\vec{r}_l)=\lambda_m v_m(\vec{r}_l) \tag{16}$$

We will now show that periodic boundary conditions, $v_m(\vec{r}_l + N_i\vec{a}_i) = v_m(\vec{r}_l)$, require that $v_m(\vec{r}_l)$ have the following form

$$v_m(\vec{r}_l) = Ae^{i\vec{k}_m\cdot\vec{r}_l} \tag{17}$$

We set the vector $\vec{k}_m$ equal to

$$\vec{k}_m = \frac{m_1}{N_1}\vec{b}_1 + \frac{m_2}{N_2}\vec{b}_2 + \frac{m_3}{N_3}\vec{b}_3 \tag{18}$$

where $m_i = 0,1,2,\ldots,N_i-1$ and the vectors $\vec{b}_i$ are reciprocal lattice vectors equal to

$$\vec{b}_i = \frac{2\pi}{a}\hat{x}_i \tag{19}$$

so that we have

$$\vec{b}_i \cdot \vec{a}_j = 2\pi\delta(i,j) \tag{20}$$

With this definition of $\vec{k}_m$ it is easy to show that eq. 17 obeys the periodic boundary conditions

$$\begin{aligned} v_m(\vec{r}_l + N_i\vec{a}_i) &= Ae^{i\vec{k}_m\cdot\vec{r}_l}e^{i\vec{k}_m\cdot N_i\vec{a}_i} = Ae^{i\vec{k}_m\cdot\vec{r}_l}e^{i2\pi m_i} \\ &= v_m(\vec{r}_l) \end{aligned} \tag{21}$$

The constant A is chosen so that the eigenvector is normalized.

$$\begin{aligned} \langle v_m|v_m\rangle &= \sum_l \langle v_m|l\rangle\langle l|v_m\rangle \\ &= \sum_l Ae^{-i\vec{k}_m\cdot\vec{r}_l}Ae^{i\vec{k}_m\cdot\vec{r}_l} \\ &= A^2N_1N_2N_3 \end{aligned} \tag{22}$$

therefore let $A = 1/\sqrt{N_1N_2N_3}$. We can now find the eigenvalues by substituting eq. 17 into eq. 16. This gives

$$\begin{aligned} \lambda_m &= 2\left(\cos\vec{k}_m\cdot\vec{a}_1 + \cos\vec{k}_m\cdot\vec{a}_2 + \cos\vec{k}_m\cdot\vec{a}_3 - 3\right) \\ &= 2\left(\cos\frac{2\pi m_1}{N_1} + \cos\frac{2\pi m_2}{N_2} + \cos\frac{2\pi m_3}{N_3} - 3\right) \end{aligned} \tag{23}$$

Now that the eigenvalue problem has been solved we can express L and $G = -L^{-1}$ in terms of the eigenbasis. For L we have

$$L = \sum_m \lambda_m |v_m\rangle\langle v_m| \tag{24}$$

The matrix elements of L are then

$$\begin{aligned} L_{ln} &= \sum_m \lambda_m \langle l|v_m\rangle\langle v_m|n\rangle \\ &= \frac{1}{N_1N_2N_3}\sum_m \lambda_m e^{i\vec{k}_m\cdot(\vec{r}_l-\vec{r}_n)} \end{aligned} \tag{25}$$

It is not too difficult to show that this equation gives the same results as in eq. 9. For $G = -L^{-1}$ we have

$$G = -\sum_m \frac{|v_m\rangle\langle v_m|}{\lambda_m} \tag{26}$$

and the matrix elements are

$$G_{ln} = -\sum_m \frac{\langle l|v_m\rangle\langle v_m|n\rangle}{\lambda_m} = -\frac{1}{N_1N_2N_3}\sum_m \frac{e^{i\vec{k}_m\cdot(\vec{r}_l-\vec{r}_n)}}{\lambda_m} \tag{27}$$

Note that G_{ln} depends only on the difference $\vec{r}_l - \vec{r}_n$ so that G has a circulant matrix representation. Let $\vec{r}_p = \vec{r}_l - \vec{r}_n$ then

$$\begin{aligned} \vec{r}_p &= (l_1 - n_1)\vec{a}_1 + (l_2 - n_2)\vec{a}_2 + (l_3 - n_3)\vec{a}_3 \\ &= p_1\vec{a}_1 + p_2\vec{a}_2 + p_3\vec{a}_3 \end{aligned} \tag{28}$$

and

$$\vec{k}_m \cdot \vec{r}_p = 2\pi \frac{m_1 p_1}{N_1} + 2\pi \frac{m_2 p_2}{N_2} + 2\pi \frac{m_3 p_3}{N_3} \tag{29}$$

Using this notation, eq. 27 becomes

$$G_{ln} = G(\vec{r}_p) = \frac{1}{N_1 N_2 N_3} \sum_{m_1=0}^{N_1-1} \sum_{m_2=0}^{N_2-1} \sum_{m_3=0}^{N_3-1} \frac{e^{i\frac{2\pi m_1 p_1}{N_1}} e^{i\frac{2\pi m_2 p_2}{N_2}} e^{i\frac{2\pi m_3 p_3}{N_3}}}{2\left(3 - \cos\frac{2\pi m_1}{N_1} - \cos\frac{2\pi m_2}{N_2} - \cos\frac{2\pi m_3}{N_3}\right)} \tag{30}$$

Eq. 30 gives the matrix elements of the lattice Green function of the DPE for a finite lattice with periodic boundary conditions. Note that this is essentially a Fourier series expansion of the matrix elements which is possible because of the periodic boundary conditions. For other boundary conditions such as $G(N_i\vec{a}_i) = 0$, the expansion would have to be in terms of a sine series.

We will now go from a finite lattice to an infinite lattice by letting $N_i \to \infty$, $i = 1, 2, 3$. This means going from the Fourier series representation of eq. 30 to a Fourier transform representation of the matrix elements. In eq. 30 let

$$x_i = \frac{2\pi m_i}{N_i} \tag{31}$$

When m_i is incremented by 1 the change in x_i is

$$\triangle x_i = \frac{2\pi}{N_i} \quad \text{or} \quad \frac{1}{N_i} = \frac{\triangle x_i}{2\pi} \tag{32}$$

The summations in eq. 30 can then be written as

$$\frac{1}{N_i} \sum_{m_i=0}^{N_i-1} \equiv \sum_{x_i=0}^{2\pi(1-\frac{1}{N_i})} \frac{\triangle x_i}{2\pi} \tag{33}$$

and in the limit $N_i \to \infty$ the summation becomes an integral.

$$\frac{1}{2\pi} \int_0^{2\pi} dx_i \tag{34}$$

For an infinite lattice eq. 30 then becomes

$$G(\vec{r}_p) = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{ix_1 p_1} e^{ix_2 p_2} e^{ix_3 p_3}}{2(3 - \cos x_1 - \cos x_2 - \cos x_3)} dx_1 dx_2 dx_3 \tag{35}$$

Note that the integrand has a period of 2π in each of the variables so that the limits of integration can be changed to the more symmetric

$$G(\vec{r}_p) = \frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{ix_1p_1}e^{ix_2p_2}e^{ix_3p_3}}{2(3-\cos x_1-\cos x_2-\cos x_3)}dx_1dx_2dx_3 \tag{36}$$

The integral can be further simplified by looking at the parity properties of the integrand. Multiplying the $e^{ix_ip_i} = \cos x_ip_i + i\sin x_ip_i$ factors and leaving out the resulting odd terms reduces the integral to

$$G(p_1,p_2,p_3) = \frac{1}{2\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{\cos x_1p_1\cos x_2p_2\cos x_3p_3}{3-\cos x_1-\cos x_2-\cos x_3}dx_1dx_2dx_3 \tag{37}$$

Clearly G is a function only of the parameters p_1, p_2, and p_3 and it is an even function of these parameters. G is also symmetric under any permutation of the parameters. All the unique values of G are therefore contained in the wedge $p_1 \geq p_2 \geq p_3 \geq 0$.

We will now derive a recurrence equation that the matrix elements of G obey. By definition we have $LG = -I$ which in the lattice basis is

$$\begin{aligned}\sum_n \langle l|L|n\rangle\langle n|G|m\rangle &= -\langle l|m\rangle \\ \sum_n L_{ln}G_{nm} &= -\delta(l,m) \\ \sum_n L(\vec{r}_l-\vec{r}_n)G(\vec{r}_n-\vec{r}_m) &= -\delta(l,m)\end{aligned} \tag{38}$$

Substituting in eq. 10 for L_{ln} gives

$$-6G(\vec{r}_l-\vec{r}_m)+\sum_{i=1}^{3}[G(\vec{r}_l+\vec{a}_i-\vec{r}_m)+G(\vec{r}_l-\vec{a}_i-\vec{r}_m)] = -\delta(l,m) \tag{39}$$

Now using the notation, $\vec{r}_l-\vec{r}_m = (l_1-m_1)\vec{a}_1+(l_2-m_2)\vec{a}_2+(l_3-m_3)\vec{a}_3 = p_1\vec{a}_1+p_2\vec{a}_2+p_3\vec{a}_3$, eq. 39 becomes

$$-6G(p_1,p_2,p_3)+G(p_1+1,p_2,p_3)+G(p_1-1,p_2,p_3)+G(p_1,p_2+1,p_3)+G(p_1,p_2-1,p_3)+G(p_1,p_2,p_3+1)+ \tag{40}$$

Eq. 40 simplifies considerably for some specific values of p_1, p_2, and p_3. In particular for $p_1 = p_2 = p_3 = 0$ we get

$$G(1,0,0) = G(0,0,0)-\frac{1}{6} \tag{41}$$

Where the symmetry properties of G have been used, i.e. $G(1,0,0) = G(-1,0,0) = G(0,1,0) = G(0,-1,0) = G(0,0,1) = G(0,0,-1)$. Letting $p_1 = p_2 = p_3 = p$ in eq. 40, we have

$$2G(p,p,p) = G(p+1,p,p)+G(p,p,p-1) \tag{42}$$

Letting $p_1 = p$, $p_2 = p_3 = 0$ in eq. 40 gives

$$G(p+1,0,0) = 6G(p,0,0) - 4G(p,1,0) - G(p-1,0,0) \tag{43}$$

Letting $p_1 = p_2 = p$, $p_3 = 0$ in eq. 40 gives

$$3G(p,p,0) = G(p+1,p,0) + G(p,p-1,0) + G(p,p,1) \tag{44}$$

In general for $p_1 = l$, $p_2 = m$, $p_3 = n$ with l, m, and n not all equal to zero, we have

$$6G(l,m,n) = G(l+1,m,n) + G(l-1,m,n) + G(l,m+1,n) + G(l,m-1,n) + G(l,m,n+1) + G(l,m,n-1) \tag{45}$$

Additional recursion equations were developed by Duffin and Shelly. These recursion equations, along with some of those given above and some relations due to Horiguchi and Morita, allowed Glasser and Boersma to find an expression for the general matrix element $G(l,m,n)$ that involves knowing only $G(0,0,0)$, which is given by the integral

$$G(0,0,0) = \frac{1}{2\pi^3} \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \frac{dx_1 dx_2 dx_3}{3 - \cos x_1 - \cos x_2 - \cos x_3} \tag{46}$$

This integral was first evaluated by Watson in terms of complete elliptic integrals. It was then shown by Glasser and Zucker to be expressible in terms of gamma functions as

$$G(0,0,0) = \frac{\sqrt{6}}{96\pi^3} \Gamma\left(\frac{1}{24}\right) \Gamma\left(\frac{5}{24}\right) \Gamma\left(\frac{7}{24}\right) \Gamma\left(\frac{11}{24}\right) \tag{47}$$

An identity due to Borwein and Zucker allows this to be simplified to

$$G(0,0,0) = \frac{\sqrt{3}-1}{96\pi^3} \Gamma^2\left(\frac{1}{24}\right) \Gamma^2\left(\frac{11}{24}\right) \tag{48}$$

Joyce [8] has also developed some recursion equations that allow $G(l,m,n)$ to be calculated for arbitrary values of l,m,n. He arrives at the same formula as Glasser and Boersma via a different method and also derives an asymptotic formula for $G(l,m,n)$. In some very recent work, Joyce [9] gives some formulas that allow the diagonal elements, $G(n,n,n)$, to be calculated very accurately for arbitrary values of n. He also gives asymptotic formulas for $G(n,n,n)$.

III. TWO DIMENSIONAL DISCRETE POISSON EQUATION

The same procedure given above can be used to find the lattice Green function for the two dimensional Poisson equation. In this case the lattice vectors are

$$\vec{r}_n = n_1 \vec{a}_1 + n_2 \vec{a}_2 \tag{49}$$

The matrix elements of the lattice Laplacian are

$$L_{ln} = \begin{cases} -4\delta(\vec{r}_l, \vec{r}_n) & \\ 1 & \text{if } |\vec{r}_n - \vec{r}_l| = a \\ 0 & \text{otherwise} \end{cases} \tag{50}$$

Which can also be expressed as

$$L_{ln} = -4\delta(\vec{r_l}, \vec{r}_n) + \delta(\vec{r}_l + \vec{a}_1, \vec{r}_n) + \delta(\vec{r}_l - \vec{a}_1, \vec{r}_n) + \delta(\vec{r}_l + \vec{a}_2, \vec{r}_n) + \delta(\vec{r}_l - \vec{a}_2, \vec{r}_n) \tag{51}$$

The eigenvector expansion of the lattice Laplacian matrix elements are

$$L_{ln} = \frac{1}{N_1 N_2} \sum_m \lambda_m e^{i\vec{k}_m \cdot (\vec{r}_l - \vec{r}_n)} \tag{52}$$

$$\vec{k}_m = \frac{m_1}{N_1}\vec{b}_1 + \frac{m_2}{N_2}\vec{b}_2 \qquad m_i = \text{integer} \tag{53}$$

$$\begin{aligned} \lambda_m &= 2\left(\cos\vec{k}_m \cdot \vec{a}_1 + \cos\vec{k}_m \cdot \vec{a}_2 - 2\right) \\ &= 2\left(\cos\frac{2\pi m_1}{N_1} + \cos\frac{2\pi m_2}{N_2} - 2\right) \end{aligned} \tag{54}$$

The matrix elements of the lattice Green function are expanded in the eigenbasis as

$$G_{ln} = -\frac{1}{N_1 N_2} \sum_m \frac{e^{i\vec{k}_m \cdot (\vec{r}_l - \vec{r}_n)}}{\lambda_m} \tag{55}$$

which if we let $\vec{r}_p = \vec{r}_l - \vec{r}_n$, can be expressed as

$$G_{ln} = G(\vec{r}_p) = \frac{1}{N_1 N_2} \sum_{m_1=0}^{N_1-1} \sum_{m_2=0}^{N_2-1} \frac{e^{i\frac{2\pi m_1 p_1}{N_1}} e^{i\frac{2\pi m_2 p_2}{N_2}}}{2\left(2 - \cos\frac{2\pi m_1}{N_1} - \cos\frac{2\pi m_2}{N_2}\right)} \tag{56}$$

For the infinite lattice this becomes

$$G(\vec{r}_p) = G(p_1, p_2) = \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \frac{\cos x_1 p_1 \cos x_2 p_2}{2 - \cos x_1 - \cos x_2} dx_1 dx_2 \tag{57}$$

G is an even function of the parameters $p_1$and p_2, and it is symmetric under any permutation of the parameters. All the unique values of G are therefore contained in the wedge $p_1 \geq p_2 \geq 0$.

There is one problem with eq. 57. The integral is divergent for all values of p_1 and p_2. We can fix this by using the shifted Green function.

$$g(p_1, p_2) = G(0,0) - G(p_1, p_2) = \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \frac{1 - \cos x_1 p_1 \cos x_2 p_2}{2 - \cos x_1 - \cos x_2} dx_1 dx_2 \tag{58}$$

The integral now exists for all values of p_1 and p_2. Using $g(p_1, p_2)$ instead of $G(p_1, p_2)$ will provide a solution to the DPE as long as the sum of the source terms, $f(\vec{r}_n)$, over all the lattice sites is equal to zero. To demonstrate this, first note that the solution to the DPE in terms of G is given by

$$\phi(\vec{r}_l) = -\sum_n G_{ln} f(\vec{r}_n) \tag{59}$$

Now if we have

$$\sum_n f(\vec{r}_n) = 0 \tag{60}$$

then eq. 59 can also be written as

$$\phi(\vec{r}_l) = \sum_n (G_{ll} - G_{ln}) f(\vec{r}_n) \tag{61}$$

where $G_{ll} = G(\vec{r}_l - \vec{r}_l) = G(0,0)$, $G_{ln} = G(\vec{r}_l - \vec{r}_n) = G(p_1, p_2)$ and $G_{ll} - G_{ln} = g_{ln}$. The solution to the DPE in terms of the shifted Green function is then

$$\phi(\vec{r}_l) = \sum_n g_{ln} f(\vec{r}_n) \tag{62}$$

where $g_{ln} = g(\vec{r}_l - \vec{r}_n) = g(p_1, p_2) = G(0,0) - G(p_1, p_2)$.

From the above discussion, you can see that in an unbounded two dimensional space or lattice the DPE is only solvable if the sources add up to zero. A physical example of this is in two dimensional electrostatics. The charge units in two dimensional electrostatics are actually parallel, infinite line charges embedded in a three dimensional space. For a single line charge, the potential at any finite distance from the line will be infinite. For two lines of opposite charge the potential is finite in the space surrounding the lines. Note that we are assuming an unbounded space with the zero point potential at infinity. Another example comes from the theory of random walks. In one and two dimensions a random walker is guarranteed to eventually return to its starting position, while in three dimensions it may never do so. To see how this is related to the DPE, see the excellent book by Doyle and Snell [10] on random walks in electrical networks. For another example see Cserti's paper [4] on using the lattice Green function to calculate the resistance between two points in an infinite network of resistors.

We will now present some recurrence equations that the matrix elements of the Green function obey. As in the three dimensional case these can easily be found from the defining relation $LG = -I$. This gives the general recurrence

$$-4G(p_1, p_2) + G(p_1+1, p_2) + G(p_1-1, p_2) + G(p_1, p_2+1) + G(p_1, p_2-1) = -\delta(p_1, 0)\delta(p_2, 0) \tag{63}$$

For $p_1 = p_2 = 0$ we have

$$G(0,0) - G(1,0) = \frac{1}{4} \tag{64}$$

For $p_1 = p \neq 0$, $p_2 = 0$ we have

$$4G(p,0) = G(p+1,0) + G(p-1,0) + 2G(p,1) \tag{65}$$

For $p_1 = p_2 = p \neq 0$ we have

$$2G(p,p) = G(p+1,p) + G(p,p-1) \tag{66}$$

And in general for $p_1 = l \neq 0$ and $p_2 = m \neq 0$ we have

$$4G(l,m) = G(l+1,m) + G(l-1,m) + G(l,m+1) + G(l,m-1) \tag{67}$$

An additional recurrence equation for the diagonal elements is [11]

$$(2n+1)G(n+1,n+1) - 4nG(n,n) + (2n-1)G(n-1,n-1) = 0 \tag{68}$$

Since the coefficients in each of these equations adds to zero you can see that the shifted Green function, $g(p_1,p_2) = G(0,0) - G(p_1,p_2)$ must obey the same recurrence equations. These equations for g are listed below

$$g(1,0) = \frac{1}{4} \tag{69}$$

$$4g(p,0) = g(p+1,0) + g(p-1,0) + 2g(p,1) \tag{70}$$

$$2g(p,p) = g(p+1,p) + g(p,p-1) \tag{71}$$

$$4g(l,m) = g(l+1,m) + g(l-1,m) + g(l,m+1) + g(l,m-1) \tag{72}$$

$$(2n+1)g(n+1,n+1) - 4ng(n,n) + (2n-1)g(n-1,n-1) = 0 \tag{73}$$

Acknowledgments

Thanks.

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