Walks on Infinite Lattices

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Linear Lattice

For an infinite linear lattice how many walks of a given length l are there between the origin and node a? Let p be the number of steps the walk takes in the positive direction (increasing node numbers) and n be the number of steps the walk takes in the negative direction (decreasing node numbers). To end up at node a, the following must be true: $p - n = a$ and $l = p + n = 2n + a$. The number of possible walks can then be written as:

$$N = \frac{l!}{n!p!} = \frac{(2n+a)!}{n!(n+a)!} \tag{1}$$

$$N = \binom{2n+a}{n} \tag{2}$$

This expression gives the total number of possible walks between node 0 and node a with length $2n + a$ where n is a non-negative integer.

Square Lattice

For an infinite square lattice how many walks of a given length l are there between the origin $(0, 0)$ and node (a, b) ? Let p_i and n_i be the number of steps in the positive and negative i^{th}directions. To end up at node (a, b), the following must be true: $p_1 - n_1 = a$, $p_2 - n_2 = b$ and $l = p_1 + n_1 + p_2 + n_2 = 2(n_1 + n_2) + a + b = 2n + a + b$ where $n = n_1 + n_2$. Now rephrase the question slightly to ask how many walks are there of length $l = 2n + a + b$ where n is some non-negative integer. The answer is simply the sum over all possible values of n_1 and n_2 such that $n_1 + n_2 = n$.

$$N = \sum_{n_1+n_2=n} \frac{(2n+a+b)!}{n_1!(n_1+a)!n_2!(n_2+b)!} \tag{3}$$

using $n_2 = n - n_1$, this can be written as

$$N = \sum_{n_1=0}^{n} \frac{(2n+a+b)!}{n_1!(n_1+a)!(n-n_1)!(n-n_1+b)!} \tag{4}$$

and with some rearranging of terms this becomes

$$N = \frac{(2n+a+b)!}{n!(n+a+b)!} \sum_{n_1=0}^{n} \frac{n!(n+a+b)!}{n_1!(n_1+a)!(n-n_1)!(n-n_1+b)!} \tag{5}$$

$$N=\binom{2n+a+b}{n}\sum_{n_1=0}^{n}\binom{n}{n_1}\binom{n+a+b}{n_1+a} \tag{6}$$

This can be simplified using the following identity:

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+a}{k+b}=\binom{2n+a}{n+b} \tag{7}$$

The expression for N then becomes

$$N=\binom{2n+a+b}{n}\binom{2n+a+b}{n+a} \tag{8}$$

This expression gives the total number of possible walks between node $(0,0)$ and node (a,b) with length $2n+a+b$ where n is a non-negative integer.

Cubic Lattice

For an infinite cubic lattice how many walks of a given length l are there between the origin $(0,0,0)$ and node (a,b,c) ? Let p_i and n_i be the number of steps in the positive and negative i^{th}directions. To end up at node (a,b,c), the following must be true: $p_1-n_1=a$, $p_2-n_2=b$, $p_3-n_3=c$ and $l=p_1+n_1+p_2+n_2+p_3+n_3=2(n_1+n_2+n_3)+a+b+c=2n+a+b+c$ where $n=n_1+n_2+n_3$. Now rephrase the question slightly to ask how many walks are there of length $l=2n+a+b+c$ where n is some non-negative integer. The answer is simply the sum over all possible values of n_1, n_2 and n_3 such that $n_1+n_2+n_3=n$.

$$N=\sum_{n_1+n_2+n_3=n}\frac{(2n+a+b+c)!}{n_1!(n_1+a)!n_2!(n_2+b)!n_3!(n_3+c)!} \tag{9}$$

Now with $n_1$ranging from 0 to n, n_2 can range from 0 to $n-n_1$and $n_3=n-n_1-n_2$ so N can also be written as

$$N=\sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\frac{(2n+a+b+c)!}{n_1!(n_1+a)!n_2!(n_2+b)!(n-n_1-n_2)!(n-n_1-n_2+c)!} \tag{10}$$

with some rearranging of terms this becomes

$$N=\binom{2n+a+b+c}{n}\sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\frac{n!(n+a+b+c)!}{n_1!(n_1+a)!n_2!(n_2+b)!(n-n_1-n_2)!(n-n_1-n_2+c)!} \tag{11}$$

$$N=\binom{2n+a+b+c}{n}\sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n+a+b+c}{n_1+a}\binom{n-n_1+b+c}{n_2+b} \tag{12}$$

$$N=\binom{2n+a+b+c}{n}\sum_{n_1=0}^{n}\binom{n}{n_1}\binom{n+a+b+c}{n_1+a}\sum_{n_2=0}^{n-n_1}\binom{n-n_1}{n_2}\binom{n-n_1+b+c}{n_2+b} \tag{13}$$

The inner summation can be simplified using the identity in eq. 7 to give

$$N=\binom{2n+a+b+c}{n}\sum_{n_1=0}^{n}\binom{n}{n_1}\binom{n+a+b+c}{n_1+a}\binom{2(n-n_1)+b+c}{n-n_1+b} \tag{14}$$

which is equivalent to

$$N = \binom{2n+a+b+c}{n} \sum_{n_1=0}^{n} \binom{n}{n_1} \binom{n+a+b+c}{n_1+b+c} \binom{2n_1+b+c}{n_1+b} \tag{15}$$

now change the notation by letting $k = n_1$ and finally we get

$$N = \binom{2n+a+b+c}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{n+a+b+c}{k+b+c} \binom{2k+b+c}{k+b} \tag{16}$$

This expression gives the total number of possible walks between node $(0,0,0)$ and node (a,b,c) with length $2n+a+b+c$ where n is a non-negative integer.