A Recursive Formula for Sampling a Sine Wave

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Suppose we want to sample $y(t) = \sin \omega t$ at intervals of T so that

$$y_n = y(nT) = \sin nT\omega$$
$$n = 0, 1, 2, \dots$$

$f_s = 1/T$ is the sampling frequency so that

$$T\omega = T2\pi f = 2\pi f/f_s$$

Call $\Omega = 2\pi f/f_s$ a dimensionless frequency, so that

$$y_n = \sin n\Omega$$

Calculating sine functions is computationally expensive, so we don't want to have to calculate $\sin n\Omega$ for every new value of n. Luckily, we don't have to. There is a recursive formula for calculating y_n from y_{n-1} and y_{n-2}. To derive this formula, start by letting

$$x_n = \cos n\Omega$$

Then we can define a complex sequence z_n as follows

$$\begin{aligned} z_n &= x_n + jy_n = e^{jn\Omega} = e^{j\Omega}e^{j(n-1)\Omega} = e^{j\Omega}z_{n-1} \\ &= (\cos\Omega + j\sin\Omega)(x_{n-1} + jy_{n-1}) \\ &= \cos\Omega\ x_{n-1} - \sin\Omega\ y_{n-1} + j(\sin\Omega\ x_{n-1} + \cos\Omega\ y_{n-1}) \end{aligned}$$

The real part of z_n is

$$x_n = \cos\Omega\ x_{n-1} - \sin\Omega\ y_{n-1}$$

and the imaginary part is

$$y_n = \sin\Omega\ x_{n-1} + \cos\Omega\ y_{n-1}$$

Solve the last expression for x_{n-1}

$$x_{n-1} = \frac{y_n - \cos\Omega\ y_{n-1}}{\sin\Omega}$$

Substitute this into the last expression for x_n

$$x_n = \frac{\cos\Omega}{\sin\Omega}(y_n - \cos\Omega\ y_{n-1}) - \sin\Omega\ y_{n-1}$$

Subtract one from the index of the last equation

$$x_{n-1} = \frac{\cos\Omega}{\sin\Omega}(y_{n-1} - \cos\Omega\ y_{n-2}) - \sin\Omega\ y_{n-2}$$

Substitute this equation into the last expression for y_n and simplify to get the following recursive equation

$$y_n = 2\cos\Omega\ y_{n-1} - y_{n-2}$$

To use this equation you just have to calculate $\cos\Omega$ one time. The formula is an efficient way to generate sine wave samples.