Chebyshev Polynomials

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1 Polynomials of the First Kind

There are many ways to define the Chebyshev polynomials. We will define the Chebyshev polynomials of the first kind as solutions to the following recurrence equation.

$$\begin{aligned} T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) &= 0 \\ T_0(x) &= 1 \\ T_1(x) &= x \end{aligned}$$

The following table lists the first 13 polynomials along with their factorization over the integers.

n	$T_n(x)$	Factored
0	1	1
1	x	x
2	$2x^2-1$	$2x^2-1$
3	$4x^3-3x$	$x\left(4x^2-3\right)$
4	$8x^4-8x^2+1$	$8x^4-8x^2+1$
5	$16x^5-20x^3+5x$	$x\left(16x^4-20x^2+5\right)$
6	$32x^6-48x^4+18x^2-1$	$\left(2x^2-1\right)\left(16x^4-16x^2+1\right)$
7	$64x^7-112x^5+56x^3-7x$	$x\left(64x^6-112x^4+56x^2-7\right)$
8	$128x^8-256x^6+160x^4-32x^2+1$	$128x^8-256x^6+160x^4-32x^2+1$
9	$256x^9-576x^7+432x^5-120x^3+9x$	$x\left(4x^2-3\right)\left(64x^6-96x^4+36x^2-3\right)$
10	$512x^{10}-1280x^8+1120x^6-400x^4+50x^2-1$	$\left(2x^2-1\right)\left(256x^8-512x^6+304x^4-48x^2+1\right)$
11	$1024x^{11}-2816x^9+2816x^7-1232x^5+220x^3-11x$	$x\left(1024x^{10}-2816x^8+2816x^6-1232x^4+220x^2-11\right)$
12	$2048x^{12}-6144x^{10}+6912x^8-3584x^6+840x^4-72x^2+1$	$\left(8x^4-8x^2+1\right)\left(256x^8-512x^6+320x^4-64x^2+1\right)$

The recurrence equation can be solved in closed form. This gives the following formula for $T_n(x)$

$$T_n(x) = \frac{z_1^n + z_2^n}{2}$$

$$\begin{aligned} z_1 &= x + \sqrt{x^2 - 1} \\ z_2 &= x - \sqrt{x^2 - 1} \end{aligned}$$

where z_1 and z_2 satisfy the following relationships

$$z_1 z_2 = 1$$

$$z_1 + z_2 = 2x$$

$$z_1 - z_2 = 2\sqrt{x^2 - 1}$$

These relationships can be useful in deriving properties of the $T_n(x)$. If $|x| \leq 1$ then we can set $x = \cos\theta$ and z_1 and z_2 become

$$z_1 = \cos\theta + i \sin\theta = e^{i\theta}$$

$$z_2 = \cos\theta - i \sin\theta = e^{-i\theta}$$

So that for $x = \cos\theta$ we have

$$T_n(x) = \frac{e^{in\theta} + e^{-in\theta}}{2} = \cos n\theta$$

Using this form for $T_n(x)$ it is very easy to derive the orthogonality relationship for these polynomials. Starting with the following well known integral

$$\int_0^\pi \cos m\theta \cos n\theta d\theta = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \frac{\pi}{2} & m = n \neq 0 \end{cases}$$

and making the substitution $x = \cos\theta$ we get

$$\int_{-1}^1 \frac{T_m(x) T_n(x)}{\sqrt{1 - x^2}} dx = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \frac{\pi}{2} & m = n \neq 0 \end{cases}$$

The $T_n(x)$ are therefore orthogonal under the weight function $w(x) = (1 - x^2)^{-\frac{1}{2}}$ however note that they are not orthonormal.

Now if $|x| > 1$ then we can set $x = \cosh\theta$ and z_1 and z_2 become

$$z_1 = \cosh\theta + \sinh\theta = e^{\theta}$$

$$z_2 = \cosh\theta - \sinh\theta = e^{-\theta}$$

So that for $x = \cosh\theta$ we have

$$T_n(x) = \frac{e^{n\theta} + e^{-n\theta}}{2} = \cosh n\theta$$

The $T_n(x)$ are not orthogonal outside the interval $-1 \leq x \leq 1$.

We will now look at the roots of $T_n(x)$. First note that all the roots are in the interval $-1 \leq x \leq 1$ and they can therefore easily be found using the expression $T_n(x) = \cos n\theta$ with $x = \cos\theta$. The roots are at

$$\theta_k = \frac{(2k-1)\pi}{2n} \qquad k = 1, 2, \ldots n$$

or

$$x_k = \cos\theta_k = \cos\frac{(2k-1)\pi}{2n} \qquad k = 1, 2, \ldots n$$

With the roots in hand, we can write $T_n(x)$ in product form as

$$T_n(x) = 2^{n-1} \prod_{k=1}^{n} (x - x_k)$$

Other special values of $T_n(x)$ which are easily derived from the above relations are

$$\begin{aligned} T_n(-x) &= (-1)^n T_n(x) \\ T_n(1) &= 1 \\ T_n(-1) &= (-1)^n \\ T_{2n}(0) &= (-1)^n \\ T_{2n-1}(0) &= 0 \end{aligned}$$

It is possible to extend the definition of the $T_n(x)$ to negative index.

$$T_{-n}(x) = \frac{z_1^{-n} + z_2^{-n}}{2} = \frac{z_2^n + z_1^n}{2} = T_n(x)$$

$T_n(x)$ is therefore even with respect to the index.

The formula for the product of two polynomials is easy to derive using the trigonometric form

$$T_n(x)T_m(x) = \cos n\theta \cos m\theta = \frac{\cos(n+m)\theta + \cos(n-m)\theta}{2} = \frac{T_{n+m}(x) + T_{n-m}(x)}{2}$$

The polynomials can also be combined iteratively as follows

$$T_n(T_m(x)) = T_n(\cos m\theta) = \cos nm\theta = T_{nm}(x)$$

This property can be used to explain some of the patterns in the factorization of the polynomials that can be seen in the table above.

The $T_n(x)$ can also be defined by the following generating function

$$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} T_n(x) t^n$$

2 Polynomials of the Second Kind

We will define the Chebyshev polynomials of the second kind as solutions to the following recurrence equation.

$$\begin{aligned} U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) &= 0 \\ U_0(x) &= 1 \\ U_1(x) &= 2x \end{aligned}$$

The following table lists the first 13 polynomials along with their factorization over the integers.

n	$U_n(x)$	Factored
0	1	1
1	$2x$	$2x$
2	$4x^2-1$	$(2x-1)(2x+1)$
3	$8x^3-4x$	$4x\left(2x^2-1\right)$
4	$16x^4-12x^2+1$	$\left(4x^2-2x-1\right)\left(4x^2+2x-1\right)$
5	$32x^5-32x^3+6x$	$2x(2x-1)(2x+1)\left(4x^2-3\right)$
6	$64x^6-80x^4+24x^2-1$	$\left(8x^3-4x^2-4x+1\right)\left(8x^3+4x^2-4x-1\right)$
7	$128x^7-192x^5+80x^3-8x$	$8x\left(2x^2-1\right)\left(8x^4-8x^2+1\right)$
8	$256x^8-448x^6+240x^4-40x^2+1$	$(2x-1)(2x+1)\left(8x^3-6x-1\right)\left(8x^3-6x+1\right)$
9	$512x^9-1024x^7+672x^5-160x^3+10x$	$2x\left(4x^2-2x-1\right)\left(4x^2+2x-1\right)\left(16x^4-20x^2+5\right)$
10	$1024x^{10}-2304x^8+1792x^6-560x^4+60x^2-1$	$\left(32x^5-16x^4-32x^3+12x^2+6x-1\right)$ $\left(32x^5+16x^4-32x^3-12x^2+6x+1\right)$
11	$2048x^{11}-5120x^9+4608x^7-1792x^5+280x^3-12x$	$4x(2x-1)(2x+1)\left(2x^2-1\right)\left(4x^2-3\right)\left(16x^4-16x^2+1\right)$
12	$4096x^{12}-11264x^{10}+11520x^8-5376x^6+1120x^4-84x^2+1$	$\left(64x^6-32x^5-80x^4+32x^3+24x^2-6x-1\right)$ $\left(64x^6+32x^5-80x^4-32x^3+24x^2+6x-1\right)$

The recurrence equation can be solved in closed form. This gives the following formula for $U_n(x)$

$$\begin{aligned} U_n(x) &= \frac{z_1^{n+1}-z_2^{n+1}}{z_1-z_2} \\ z_1 &= x+\sqrt{x^2-1} \\ z_2 &= x-\sqrt{x^2-1} \end{aligned}$$

The z's are the same as those defined for $T_n(x)$. If $|x|\le 1$ then we can set $x=\cos\theta$ so that $z_1=e^{i\theta}$, $z_2=e^{-i\theta}$ and then $U_n(x)$ becomes

$$U_n(x)=\frac{e^{i(n+1)\theta}-e^{-i(n+1)\theta}}{e^{i\theta}-e^{-i\theta}}=\frac{\sin(n+1)\theta}{\sin\theta}$$

From this trigonometric form, it is easy to derive the orthogonality relationship. First start with the integral

$$\int_0^\pi \sin(n+1)\theta\,\sin(m+1)\theta\,d\theta=\frac{\pi}{2}\delta_{n,m}$$

then make the substitution $x=\cos\theta$ to get

$$\int_{-1}^1 \sqrt{1-x^2}U_n(x)\,U_m(x)\,dx=\frac{\pi}{2}\delta_{n,m}$$

The $U_n(x)$ are orthogonal under the weight function $w(x)=(1-x^2)^{\frac{1}{2}}$, note however that they are not orthonormal. For $|x|>1$ we can set $x=\cosh\theta$ to get $z_1=e^\theta$, $z_2=e^{-\theta}$ and

$$U_n(x)=\frac{e^{(n+1)\theta}-e^{-(n+1)\theta}}{e^\theta-e^{-\theta}}=\frac{\sinh(n+1)\theta}{\sinh\theta}$$

The $U_n(x)$ are not orthogonal outside the interval $-1\le x\le 1$.

We now look at the roots of $U_n(x)$ which are all in the interval $-1 \leq x \leq 1$ so we can find them using the trigonometric form of $U_n(x)$. The roots are at

$$\theta_k = \frac{k\pi}{n+1} \qquad k = 1, 2, \ldots n$$

or

$$x_k = \cos\theta_k = \cos\frac{k\pi}{n+1} \qquad k = 1, 2, \ldots n$$

We can then write $U_n(x)$ in product form as

$$U_n(x) = 2^n \prod_{k=1}^{n} (x - x_k)$$

Other special values of $U_n(x)$ are

$$\begin{aligned} U_n(-x) &= (-1)^n U_n(x) \\ U_n(1) &= n+1 \\ U_n(-1) &= (-1)^n (n+1) \\ U_{2n}(0) &= (-1)^n \\ U_{2n-1}(0) &= 0 \end{aligned}$$

It is possible to extend the definition of the $U_n(x)$ to negative index.

$$U_{-n}(x) = \frac{z_1^{-n+1} - z_2^{-n+1}}{z_1 - z_2} = \frac{z_2^{n-1} - z_1^{n-1}}{z_1 - z_2} = -U_{n-2}(x)$$

The formula for the product of two polynomials is

$$U_n(x)U_m(x) = \frac{T_{n-m}(x) - T_{n+m+2}(x)}{2(1-x^2)}$$

The $U_n(x)$ can also be defined by the following generating function

$$\frac{1}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} U_n(x) t^n$$

3 Relationships Between $T_n(x)$ and $U_n(x)$

It is easy to derive hundreds of relationships between the $T_n(x)$ and $U_n(x)$ by using their trigonometric forms or the formulas in terms of the z's. Some of the relationships are listed below.

$$\frac{d}{dx} T_n(x) = n U_{n-1}(x)$$

$$(1 - x^2) U_{n-1}(x) = x T_n(x) - T_{n+1}(x)$$

$$T_n(x) = U_n(x) - x U_{n-1}(x)$$

$$U_{n-1}(T_m(x)) = \frac{U_{nm-1}(x)}{U_{m-1}(x)}$$

References

[1] George Arfken. *Mathematical Methods for Physicists.* Academic Press, third edition, 1985.

[2] J.C. Mason and D.C. Handscomb. *Chebyshev Polynomials.* Chapman and Hall/CRC, 2003.

[3] Theodore J. Rivlin. *Chebyshev Polynomials : From Approximation Theory to Algebra and Number Theory.* John Wiley & Sons, 1990.

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