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Fourier Transform of a Sampled Signal

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The Fourier transform (FT) of a continuous signal $s(t)$ is given by:

$$S(\omega) = \int_{-\infty}^{\infty} s(t)e^{-j\omega t}dt \tag{1}$$

Now sample the signal at equally spaced intervals T so that the n^{th} sample is equal to $s(nT)$. Mathematically, the operation of sampling can be represented as multiplication of $s(t)$ by a periodic impulse train.

$$s(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty} s(nT)\delta(t-nT) \tag{2}$$

The Fourier transform of the sampled signal is then equal to the convolution of $S(\omega)$ and the FT of the impulse train. The Fourier transform of an impulse train is an impulse train in the frequency domain.

$$FT\left\{\sum_{n=-\infty}^{\infty}\delta(t-nT)\right\} = \sum_{n=-\infty}^{\infty}\omega_0\delta(\omega-n\omega_0) \qquad \omega_0 = \frac{2\pi}{T} \tag{3}$$

The convolution is then equal to

$$\begin{aligned} S_p(\omega) &= S(\omega) * \textstyle\sum_{n=-\infty}^{\infty} \omega_0 \delta(\omega - n\omega_0) \\ &= \tfrac{\omega_0}{2\pi} \textstyle\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} S(y)\delta(\omega - y - n\omega_0) dy \\ &= \tfrac{1}{T} \textstyle\sum_{n=-\infty}^{\infty} S(\omega - n\omega_0) \end{aligned} \tag{4}$$

Where $S_p(\omega)$ has been used to denote the FT of the sampled signal. Equation 4 shows that $S_p(\omega)$ consists of copies of $S(\omega)$ placed at intervals of ω_0 up and down the frequency axis. $S_p(\omega)$ is therefore a periodic function with period equal to ω_0.

Now $S_p(\omega)$ will be calculated directly from equation 2.

$$\begin{aligned} S_p(\omega) &= \textstyle\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s(nT)\delta(t - nT)e^{-j\omega t} dt \\ &= \textstyle\sum_{n=-\infty}^{\infty} s(nT) \int_{-\infty}^{\infty} \delta(t - nT)e^{-j\omega t} dt \\ &= \textstyle\sum_{n=-\infty}^{\infty} s(nT)e^{-j\omega nT} \end{aligned} \tag{5}$$

This is recognizable as a Fourier series representation of $S_p(\omega)$ where the series coefficients are equal to the samples of $s(t)$. The fact that $S_p(\omega)$ has a Fourier series representation could be anticipated since it is a periodic function.

Now let $\Omega = \omega T$ and $s[n] = s(nT)$ then equation 5 is written as

$$S_p(\Omega) = \sum_{n=-\infty}^{\infty} s[n]e^{-jn\Omega} \tag{6}$$

Ω is a dimensionless variable and $S_p(\Omega)$ now has a period of 2π, $S_p(\Omega + 2\pi) = S_p(\Omega)$. There is no longer any explicit time dependence in equation 6.

It is possible to invert equation 6 to get the sample values, $s[n]$, in terms of the FT, $S_p(\Omega)$. Multiply both sides by $e^{jm\Omega}$ and integrate over one period.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} S_p(\Omega)e^{jm\Omega} d\Omega = \sum_{n=-\infty}^{\infty} s[n] \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j(n-m)\Omega} d\Omega \tag{7}$$

The integral on the right is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j(n-m)\Omega} d\Omega = \frac{2sin((n-m)\pi)}{2\pi(n-m)} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \qquad (8)$$

Equation 7 then reduces to

$$s[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_p(\Omega) e^{jm\Omega} d\Omega \qquad (9)$$

Now use equation 6 to calculate $S_p(\Omega)$ at N equally spaced points in the interval from $\Omega = 0$ to $\Omega = 2\pi$. This amounts to sampling $S_p(\Omega)$ at intervals $\Omega_1 = 2\pi/N$. From equation 6 the k^{th} sample is then

$$S_p(k\Omega_1) = S_p(\frac{2\pi k}{N}) = \sum_{n=-\infty}^{\infty} s[n]e^{-j2\pi kn/N} \qquad (10)$$

For a given k there are only N distinct values of $e^{-j2\pi kn/N}$. This can be seen from the following equation where n is incremented to $n + mN$, $m =$ an integer.

$$e^{-j2\pi k(n+mN)/N} = e^{-j2\pi kn/N} e^{-j2\pi km} = e^{-j2\pi kn/N} \qquad (11)$$

This fact can be exploited to write equation 10 as follows.

$$\begin{aligned} S_p(k\Omega_1) &= \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} s[n+mN]e^{-j2\pi k(n+mN)/N} \\ &= \sum_{n=0}^{N-1} e^{-j2\pi kn/N} \sum_{m=-\infty}^{\infty} s[n+mN] \\ &= \sum_{n=0}^{N-1} s_p[n]e^{-j2\pi kn/N} \end{aligned} \qquad (12)$$

Where $s_p[n]$ has been defined to be

$$s_p[n] = \sum_{m=-\infty}^{\infty} s[n+mN] \qquad (13)$$

Obviously $s_p[n]$ is periodic with period N,

$$s_p[n+N] = s_p[n] \qquad (14)$$

If there are only N nonzero samples of $s[n]$ say

$s[n] \neq 0 \qquad 0 \leq n \leq N-1$
$s[n] = 0 \qquad otherwise$

then $s_p[n] = s[n],\ 0 \leq n \leq N-1$.

Equation 12 is called the discrete Fourier transform and is usually written in the following form

$$S_p[k] = \sum_{n=0}^{N-1} s_p[n] e^{-j2\pi kn/N} \qquad\qquad S_p[k] = S_p(k\Omega_1) \tag{15}$$

The effect of windowing a sampled signal

Truncating the samples $s[n]$ to just N nonzero samples can be mathematically taken into account by saying that $s[n]$ has been multiplied by the samples of a window function that has only N nonzero samples. As an example let $s(t)$ be a pure sine wave of frequency $\omega_0 = 2\pi/T_0$.

$$s(t) = sin(\omega_0 t) \tag{16}$$

Sampling at intervals T we get

$$s[n] = s(nT) = sin(\omega_0 nT) \tag{17}$$

Let $\Omega_0 = \omega_0 T$ then

$$s[n] = sin(n\Omega_0) = \frac{e^{jn\Omega_0} - e^{-jn\Omega_0}}{2j} \tag{18}$$

There are an infinite number of nonzero samples, $s[n]$. To get a finite number we multiply by the window samples $w[n]$ where $w[n] \neq 0,\ 0 \leq n \leq N-1$, and $w[n] = 0$ for all other n. Call the result of this multiplication a new sequence $q[n]$

$$q[n] = w[n]s[n] \tag{19}$$

So for $s[n]$ given by equation 18 we have

$$q[n] = w[n]\frac{e^{jn\Omega_0}}{2j} - w[n]\frac{e^{-jn\Omega_0}}{2j} \tag{20}$$

The Fourier transform of $q[n]$ is then

$$Q(\Omega) = \frac{1}{2j}FT\{w[n]e^{jn\Omega_0}\} - \frac{1}{2j}FT\{w[n]e^{-jn\Omega_0}\} \tag{21}$$

where we have

$$\begin{aligned} FT\{w[n]e^{jn\Omega_0}\} &= \textstyle\sum_{n=-\infty}^{\infty} w[n]e^{jn\Omega_0}e^{-jn\Omega} \\ &= \textstyle\sum_{n=-\infty}^{\infty} w[n]e^{-jn(\Omega-\Omega_0)} = W(\Omega-\Omega_0) \end{aligned} \tag{22}$$

likewise

$$FT\{w[n]e^{-jn\Omega_0}\} = W(\Omega+\Omega_0) \tag{23}$$

so that equation 21 becomes

$$Q(\Omega) = \frac{W(\Omega-\Omega_0) - W(\Omega+\Omega_0)}{2j} \tag{24}$$

The Fourier transform of the windowed sine wave consists of copies of the FT of the window sequence centered at Ω_0 and $-\Omega_0$.

Specific examples of window functions are discussed in a separate article.